3P17/4P17 COMPUTABILITY 2014-2015 Lecture Handout 6

Computational resources

Given a (single tape, deterministic) Turing machine M with input $\sigma \in \Sigma^*$, a sequence of configurations $C_0, C_1, \ldots$, with C_0 the starting configuration, will be generated. If the sequence ends on some halting configuration C_n, then the **time complexity** $t_M(\sigma) = n$, otherwise, it does not halt and $t_M(\sigma) = \infty$.

i.e., the time complexity is the number of steps the TM takes to halt, or infinity if it never halts.

Recalling that we number the tape locations $0, 1, 2, \ldots$ where location 0 is $\triangleright$, the **space complexity** $s_M(\sigma)$ is one less than the maximum location number the tape ever gets to in the computation of M on σ, or ∞ if there is no maximum.

i.e., the space complexity is the number of tape squares besides $\triangleright$ that the tape head of M ever reads on input σ (it does not read the symbol after the final movement, hence "one less"). We do not count the $\triangleright$ because it never changes, so we are not using that location as memory.

Note: It is possible that the time complexity is ∞ but the space complexity is finite.

Lemma 1. *For a Turing machine M with input σ over some alphabet Σ,*

$$s_M(\sigma) \leq t_M(\sigma).$$

Proof. In a sequence of configurations $C_0, C_1, \ldots, C_n$ where C_0 is the starting configuration, the maximum location reached cannot be greater than $n + 1$. □

Landau notation

Let $f, g : \mathbb{N} \to \mathbb{N}$.

We write $f(n) = O(g(n))$ if there are positive reals C, n_0 such that

$$f(n) \leq Cg(n) \qquad \forall n > n_0.$$

We write $f(n) = \Omega(n)$ if there are positive reals c, n_0 such that

$$f(n) \geq cg(n) \qquad \forall n > n_0.$$

We write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Examples

$f(n) = n^2 + 101n + 1$. Taking $C = 2$, we have $n^2 + 101n + 1 \leq 2n^2$, i.e., $101n + 1 \leq n^2$ for all, say, $n > 101$. Hence, we may take $C = 2, n_0 = 101$ to show $f(n) = O(n^2)$.

On the other hand, $n^2 + 101n + 1 \geq n^2$ for all $n \in \mathbb{N}$, so taking $c = n_0 = 1$, we see that $f(n) = \Omega(n)$.

Therefore, $f(n) = \Theta(n)$.

It can also be shown that $f(n) = O(n^3)$, $f(n) = O(e^n)$, $f(n) = O(n!)$...

It can also be shown that $f(n) = \Omega(n^{1.7})$, $f(n) = \Omega(n \log n)$, $f(n) = \Omega(1)$...

Lemma 2. *Let p_1, p_2 be polynomials of degree d_1, d_2, respectively.*

If $d_1 \leq d_2$ then $p_1(n) = O(p_2(n))$.

If $d_1 = d_2$ then $p_1 = \Theta(p_2)$.