Speeding Up Cover Time of Sparse Graphs Using Local Knowledge

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Abstract. We analyse the cover time of a random walk on a random graph of a given degree sequence. Weights are assigned to the edges of the graph using a certain type of scheme that uses only local degree knowledge. This biases the transitions of the walk towards lower degree vertices. We demonstrate that, with high probability, the cover time is at most $(1+o(1))\frac{d-1}{d-2}8n\log n$, where d is the minimum degree. This is in contrast to the precise cover time of $(1+o(1))\frac{d-1}{d-2}\frac{\theta}{d}n\log n$ (with high probability) given in [1] for a simple (i.e., unbiased) random walk on the same graph model. Here θ is the average degree and since the ratio θ/d can be arbitrarily large, or go to infinity with n, we see that the scheme can give an unbounded speed up for sparse graphs.

Keywords: Random walks, random graphs, network exploration

1 Introduction

A simple random walk $\mathcal{W}_u = \mathcal{W}_u(t)$, $t = 0, 1, \ldots$ on a graph G starting from a vertex u is a sequence of movements from one vertex to another where at each step an edge is chosen uniformly at random from the set of edges incident on the current vertex, and then transitioned to next vertex. Various quantities of interest related to the behaviour of the walk can be studied. For example, the *hitting time* $\mathbf{H}[u,v]$ of v is the expected number of steps until $\mathcal{W}_u$ visits v for the first time. That is, $\mathbf{H}[u,v] = \mathbf{E}[\min\{t \in \mathbb{N}_0 : \mathcal{W}_u(t) = v\}]$ (note, by definition, $\mathbf{H}[u,u] = 0$). The *maximum hitting time* is $\max_{u,v} \mathbf{H}[u,v]$.

Another quantity of interest, and the primary focus of this paper, is the *cover time* $\mathbf{COV}[G]$: denoting by $\mathbf{COV}_u[G]$ the expected time it takes $\mathcal{W}_u$ to visit every vertex, $\mathbf{COV}[G] = \max_u \mathbf{COV}_u[G]$.

For simple random walks, asymptotically tight bounds for cover time were given by [6] and [7]:

$$(1 + o(1))n \log n \leq \mathbf{COV}[G] \leq (1 + o(1))\frac{4}{27}n^3,$$

and these lower and upper bounds are met by the complete graph and the lollipop graph respectively.

It is also natural to define random walks on a weighted graph $G = (V, E, w)$, where w is a function mapping edges to strictly positive values $w : E \to \mathbb{R}^+$. A *weighted* random walk on a vertex u transitions an edge (u, v) with probability $w(u, v)/w(u)$. Simple random walks are a special case where w is a constant function, and we may refer to them as *unweighted* walks.

The study of random walks on general weighted graphs is less developed than the special case of unweighted graphs, and it is not difficult to formulate many open questions on their behaviour. In particular, what bounds exist for hitting times and cover time? This was addressed in part by [8] and [9]. The investigation is framed as follows. For a graph G, let $\mathcal{P}(G)$ denote the set of all transition probability matrices for G, that is, stochastic matrices that respect the graph structure, i.e. if $P \in \mathcal{P}(G)$ is a transition matrix on G, we have $P_{u,v} \neq 0$ if and only if $(u, v) \in E$.

For $P \in \mathcal{P}(G)$, let $H_G(P)$ denote the maximum hitting time in G with transition matrix P, and $C_G(P)$ similarly for cover time. Let

$$H_G = \inf_{P \in \mathcal{P}(G)} H_G(P) \quad \text{and} \quad C_G = \inf_{P \in \mathcal{P}(G)} C_G(P).$$

Note that if for a graph G one knows a spanning tree T_G, a transition matrix P can be constructed that gives a simple random walk on T_G, and ignores all other edges of G. By a "twice round the spanning tree" argument of the type employed in [3], this implies a $O(n^2)$ upper bound on H_G and C_G.

In [9], it is shown that for a path graph P_n, any transition matrix will have $\Omega(n^2)$ maximum hitting time (and therefore, cover time). This, in conjunction with the spanning tree argument, implies $\Theta(n^2)$ for both H_G and C_G.

One can also ask the question about the minimum local topological information on the graph G that is always sufficient to construct a transition matrix that 'achieves' this upper bound for both H_G and C_G. Our goal is to devise a particular weighting scheme that gives $O(n^2)$ maximum hitting time for any graph. In [9], the transition probability of edge $e = (u, v)$ is defined as follows:

$$P_{u,v} = \begin{cases} \frac{1/\sqrt{d(v)}}{\sum_{w \in N(u)} 1/\sqrt{d(w)}} & \text{if } v \in N(u) \\ 0 & \text{otherwise} \end{cases}$$

where $d(v)$ is the degree of v and $N(v)$ is the neighbour set of v.

We will refer to this as the *Ikeda* scheme. It results in an $O(n^2 \log n)$ upper bound on the cover time for any connected n-vertex graph G. The rationale behind this scheme is that, at a high degree vertex, the biased walk transition favours low degree neighbours, speeding up their exploration and addressing the shortcoming of simple random walks for which low degree nodes are hard to reach.

In the algorithmic context of graph exploration, simple random walks are generally considered to have the benefit of not requiring information beyond what is needed to choose the next edge **uar**. Generally, this implies that a token making the walk can be assumed to know the degree of the vertex it is currently on, but no more information about the structure of the graph. In the Ikeda scheme, information required in addition to the vertex degree, is the degrees of neighbouring vertices.

Whilst the speed up given by the Ikeda scheme is clear for graphs such as the lollipop, which has a cover time of $\Theta(n^3)$, it is not clear how much of an advantage it gives over the simple random walk for sparse graphs. Clearly for regular graphs there can be no difference, but what of graphs that have some variation in vertex degree but are still sparse and perhaps even fairly homogeneous? The main aim of this paper is to answer this question for a different local weighting scheme: for $G = (V, E)$, assign each edge (u, v) weight $w(u, v) = 1/\min\{d(u), d(v)\}$ (equivalently, each edge is assigned *resistance* $r(u, v) = \min\{d(u), d(v)\}$). This weighting scheme defines the following transition matrix of a weighted random walk:

$$P_{u,v} = \begin{cases} \frac{w(u,v)}{\sum_{w \in N(u)} w(u,w)} & \text{if } v \in N(u) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $w(u, v) = 1/\min\{d(u), d(v)\}$.

We may, as a matter of convenience, say that $w(u, v) = 0$ if $(u, v) \notin E$ in calculations of transition probabilities. We call this scheme the *minimum degree* (or *min-deg*) scheme. It uses limited local graph information as the Ikeda scheme and provides similar general bounds on the hitting time and cover time of $O(n^2)$ and $O(n^2 \log n)$ respectively. Additionally, however, we show that it can provided an arbitrarily large or unbounded speed up for sparse graphs.

Notation and terminology, structure of the paper

For a graph $G = (V, E)$, let $n = |V|$ and $m = |E|$. Asymptotic quantities are with respect to n. A sequence of events $(\mathcal{E}_n)_n$ on probability spaces indexed by the number of vertices n occurs *with high probability* (**whp**) if $\mathbf{Pr}(\mathcal{E}_n) \to 1$ as $n \to \infty$.

For a vertex $v \in V$, $d_v = d(v)$ is the degree of v and $N(v)$ is the set of v's neighbours. For a random walk $\mathcal{W}_u$ on a (weighted or unweighted) graph G, the stationary distribution, should it exist, is denoted by π, and $\pi_v = \pi(v)$ is the stationary probability for vertex v.

We use the phrases "weighted random walk on a graph G" and "random walk on a weighted graph G" interchangeably.

We shall introduce further notation as needed.

In the next section, we give general bounds for the min-deg scheme. In section 3, we begin by describing a class of sparse graphs of a given degree sequence. We give the constraints on the degree sequence and compare to established results for simple walks. We then analyse the walk weighted with the min-deg scheme on the same model of graph.

Due to space constraints, proofs not presented in the main text are given in the appendix.

2 General bounds for the minimum degree weighting scheme

In this section, we prove an upper bound of $O(n^2)$ and $O(n^2 \log n)$ on the hitting time and cover time, respectively, for the minimum degree scheme with transition matrix (1).

The proof of $O(n^2)$ hitting time in [9] applies the following, generally useful lemma (proof omitted).

Lemma 1. *For any connected graph G and any pair of vertices $u, v \in V$, let $\rho = (x_0, x_1, \ldots, x_\ell)$ where $x_0 = u$ and $x_\ell = v$ be a shortest path between u and v. Then*

$$\sum_{i=0}^{\ell} d(x_i) \leq 3n$$

where $d(x)$ is the degree of vertex x.

In addition, we have that

Lemma 2. *For the minimum degree scheme, let $w(G) = \sum_{u \in V} \sum_{v \in N(u)} w(u, v)$. Then*

$$n \leq w(G) \leq 2n. \tag{2}$$

Consequently,

Theorem 1. *For a graph $G = (V, E, w)$ under the min-deg weighting scheme, $\mathbf{H}[u, v] \leq 6n^2$ for any pair of vertices $u, v \in V$.*

By Matthews' technique [14], we obtain

Corollary 1. $\mathbf{COV}[G] = O(n^2 \log n)$.

The authors of [9] conjecture that their weighting scheme in fact gives an $O(n^2)$ upper bound on cover time. To our knowledge, no weighting scheme has been shown to meet an $O(n^2)$ bound on all simple, connected and undirected graphs. We believe that our weighting scheme provides a similar bound and conjecture so:

Conjecture 1. The minimum degree weighting scheme has $O(n^2)$ cover time on all graphs G.

3 Random graphs of a given degree sequence

From here on we study a sequence of random graphs on n vertices, where n goes to infinity.

Define $\mathcal{G}(\mathbf{d}_n)$ to be the set of connected simple graphs on the vertex set $V = [n]$ and with degree sequence $\mathbf{d}_n = (d_1^{(n)}, d_2^{(n)}, \ldots, d_n^{(n)})$ where $d_i^{(n)} = d(i)$ is the degree of vertex $i \in V$. Clearly, restrictions on degree sequences are required in order for the model to make sense. An obvious one is that the sum of the degrees in the sequence cannot be odd. Even then, not all degree sequences are *graphical* and not all graphical sequences can produce simple graphs. Take for example the two vertices v and w where $d_v = 3$ and $d_w = 1$. In order to study this model, we restrict the degree sequences to those which are *nice* and graphs which have nice degree sequences are termed the same. The precise definition is given below.

Let

$$\omega = \omega_n = \log \log \log n.$$

For a degree sequence $\mathbf{d}_n$, Let $d = d(\mathbf{d}_n) = d_1^{(n)}$ be the minimum, $\theta = \theta(\mathbf{d}_n)$ the average and $\Delta = \Delta(\mathbf{d}_n) = d_n^{(n)}$ the maximum of the entries in $\mathbf{d}_n$. Let $n_d = \sum_{i=1}^{n} \mathbf{1}_{\{d(i)=d\}}$, that is, the total number of entries in $\mathbf{d}_n$ with value d. We emphasise that d can grow with n – it need not be a fixed integer.

A sequence $(\mathbf{d}_n)_n$ of degree sequences is *nice* if the following conditions are satisfied: For each $\mathbf{d}_n$,

(i) $n\theta$ is even.

(ii) $d \geq 3$.

(iii) $\Delta \leq \omega^{1/4}$.

Furthermore,

(iv) for some constant $\alpha \in (0, 1]$, $n_d/n \to \alpha$ as $n \to \infty$.

Note that that this is a more restrictive definition of nice than in [1], and that a sequence of degree sequences that is nice by this definition is also nice per the definition given in [1].

Condition **(ii)** ensures the graph is connected (**whp**). Condition **(iii)** is required for our proofs in subsequent lemmas, and it has the effect of rendering redundant other conditions for nice sequences listed in [1].

To understand condition **(iv)** consider that without it, the sequence of degree sequences could result in sequences of random graphs that have wildly different cover times. As such we may not have convergence. The condition itself is fairly liberal – we do not require the that the degree sequence as a whole converges to a fixed distribution, nor even that d converges to some fixed constant.

Although this model is typically framed as a random graph, randomness here is superfluous. We assume that the graph G that the protocol acts upon is from the typical subset $\mathcal{G}'(\mathbf{d}_n)$ of the set $\mathcal{G}(\mathbf{d}_n)$ of simple graphs with nice degree sequence $\mathbf{d}_n$. As long as G has the typical properties, our analysis holds. The fact that the typical subset is almost the same size as the general set when n is large is demonstrated via the configuration model. See [1] for a detailed explanation.

Examples of nice sequences/graphs are: Any d-regular graph where $d \leq \omega^{1/4}$; a graph where a positive fraction of the vertices have bounded degree at least 3 and the rest have unbounded degree at most $\omega^{1/4}$; a truncated power-law graph with with minimal degree at least 3 and maximal degree at most $\omega^{1/4}$.

In [1], the authors prove the following asymptotic result on the cover time of simple random walks on nice graphs:

Theorem 2 ([1]). *Let $(\boldsymbol{d}_n)_n$ be nice and let G be chosen* **uar** *from $\mathcal{G}(\boldsymbol{d}_n)$. Then* **whp***,*

$$\mathbf{COV}[G] = (1 + o(1))\frac{d-1}{d-2}\frac{\theta}{d}\, n \log n, \tag{3}$$

where d is the effective minimum degree and θ is the average degree.

The *effective minimum degree* is the smallest integer d which satisfies condition **(iv)** above. It coincides with the minimum degree in our context.

We prove the following:

Theorem 3. *Let $(\boldsymbol{d}_n)_n$ be nice and let G be chosen* **uar** *from $\mathcal{G}(\boldsymbol{d}_n)$. Weight the edges of G with the min-deg weighting scheme, that is, for an edge (u, v), assign it weight $w(u, v) = 1/\min\{d(u), d(v)\}$. Denote the resulting graph G_w. Then* **whp***,*

$$\mathbf{COV}[G_w] \leq (1 + o(1))\frac{d-1}{d-2}\, 8n \log n. \tag{4}$$

where d is the minimum degree.

Note that the assumptions on the degree sequence allow for the ratio θ/d to be unbounded. As such, the ratio of the min-deg cover time to the simple cover time, that is, the *speed up*, can be unbounded.

Typical graphs Our analysis requires that graphs G taken from $\mathcal{G}(\mathbf{d}_n)$ have certain structural properties. The subset of graphs $\mathcal{G}'(\mathbf{d}_n)$ having these properties form a large proportion of $\mathcal{G}(\mathbf{d}_n)$, in fact, $|\mathcal{G}'(\mathbf{d}_n)|/|\mathcal{G}(\mathbf{d}_n)| = 1 - n^{-\Omega(1)}$ when $(\mathbf{d}_n)_n$ is nice ([1]). We term graphs in $\mathcal{G}'(\mathbf{d}_n)$ for $(\mathbf{d}_n)_n$ nice as *typical*, so a graph G drawn **uar** from $\mathcal{G}(\mathbf{d}_n)$ will be typical **whp**.

We need not list all the properties of typical graphs, but we shall use their useful consequences, amongst which are that they are connected, simple, and non-bipartite.

4 Convergence to the stationary distribution

In this section we begin with a brief overview of results on Markov chains and random walks on (weighted) graphs. For details, we refer the reader to, e.g., [2], [11] and [12].

Since a weighted random walk on $G = (V, E, w)$ is a reversible Markov chain, we can apply standard results for these types of processes. For example, if G is non-bipartite, then the walk converges to a stationary distribution π, where $\pi_u = \pi(u) = w(u)/w(G)$.

Furthermore, the rate of convergence is related to the the *absolute spectral gap* - the difference between the largest eigenvalue, 1 and second largest (in magnitude) eigenvalue λ_* of the probability transition matrix of the walk. Specifically, if $P_u^{(t)}(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$ then

$$|P_u^{(t)}(v) - \pi_v| \leq \sqrt{\frac{\pi_v}{\pi_u}}\,\lambda_*^t. \tag{5}$$

If the walk is made *lazy*, that is, if we append a looping probability of 1/2 and scale all other transition probabilities accordingly, then the largest eigenvalue remains 1 and second eigenvalue λ_2 is guaranteed to be the second largest in absolute terms. We can then apply the following result, proved independently in [10] and [13]:

Theorem 4 ([10], [13]). *Let λ_2 be the second largest eigenvalue of a reversible, aperiodic transition matrix $\boldsymbol{P}$. Then*

$$\frac{\Phi^2}{2} \leq 1 - \lambda_2 \leq 2\Phi \tag{6}$$

where Φ is the conductance.

Corollary 2.

$$|P_u^{(t)}(v) - \pi_v| \leq \sqrt{\frac{\pi_v}{\pi_u}} \left(1 - \frac{\Phi^2}{2}\right)^t. \tag{7}$$

Conductance is defined as follows:

Definition 1 (Conductance). *Let $\mathcal{M}$ be an irreducible, aperiodic Markov chain on some state space Ω. Let the stationary distribution of $\mathcal{M}$ be π with $\pi(x)$ denoting the stationary probability of $x \in \Omega$. Let P be the transition matrix for $\mathcal{M}$. For $x, y \in \Omega$ let $Q(x,y) = \pi(x)P_{x,y}$ and for sets $A, B \subseteq \Omega$, let $Q(A,B) = \sum_{x\in A, y\in B} Q(x,y)$. The* conductance *of $\mathcal{M}$ is the quantity*

$$\Phi = \Phi(\mathcal{M}) = \min_{\substack{S\subseteq\Omega \\ \pi(S)\leq 1/2}} \frac{Q(S,\overline{S})}{\pi(S)} \tag{8}$$

where $\pi(S) = \sum_{x\in S} \pi(x)$, and $\overline{S} = \Omega \setminus S$.

For a graph G weighted by function w, we write $\Phi(G_w)$ for the conductance of the weighted random walk on G.

Making the walk lazy halves the conductance and doubles the important quantity R_v, which we shall define and elaborate upon below. It also doubles the cover time.

In fact, we do not need to maintain a lazy walk all the time, but will do so only for the duration of the *mixing time* T which we define as follows:

$$T = \omega^2 \log n. \tag{9}$$

Informally, the mixing time is how long it takes for the distribution of a Markov chain to be close to the stationary distribution. After the mixing time, we can revert to the non-lazy walk. It will be seen that the lazy steps during the mixing time will have negligible impact on the asymptotic cover time, since, being polylogarithmic, it is short compared to other quantities such as hitting time which are linear in n and dominate over it.

More precisely, we show below that for most nice graphs, for any $t \geq T$

$$|P_u^{(t)}(x) - \pi_v| \leq \frac{1}{n^3}, \tag{10}$$

for any vertices u and v in G. This is a corollary of the following lemma:

Lemma 3. *Let $(\boldsymbol{d}_n)_n$ be nice and let G be chosen* **uar** *from $\mathcal{G}(\boldsymbol{d}_n)$. Let G_w be G weighted with the min-deg weighting scheme. Then $\Phi(G_w) \geq 1/(100\Delta)$* **whp***, where Δ is the maximum degree.*

We will consider the condition $\Phi(G_w) \geq 1/(100\Delta)$ to be one of the typical properties.

Corollary 3. *For a random walk on a weighted typical graph G, we have for $t \geq T$,*

$$|P_u^{(t)}(v) - \pi_v| \leq n^{-3},$$

where $P_u^{(t)}(v)$ is the probability that the minimum degree random walk is at node v at time t, given that it started at node u.

5 First visit lemma

The hitting time from the stationary distribution, $\mathbf{H}[\pi, v] = \sum_{u \in V} \pi_u \mathbf{H}[u, v]$, can be expressed as $\mathbf{H}[\pi, v] = Z_{v,v}/\pi_v$, where

$$Z_{v,v} = \sum_{t=0}^{\infty} (P_v^{(t)}(v) - \pi_w), \tag{11}$$

see e.g. [2]. For a (weighted or unweighted) random walk $\mathcal{W}_v$, starting from v define

$$R_v(T) \;=\; \sum_{t=0}^{T-1} P_v^{(t)}(v). \tag{12}$$

Thus R_v is the expected number of returns made by $\mathcal{W}_v$ to v during the mixing time, in the graph G. We note that $R_v \geq 1$, as $P_v^{(0)}(v) = 1$.

Let $D(t) = \max_{u,x} |P_u^{(t)}(x) - \pi_x|$. As $\pi_x \geq 1/n^2$ for any vertex of a simple graph, (10) implies that $D(t) \leq \pi_x$ for all $x \in V$ if $t \geq T$.

Lemma 4. *For a random walk $\mathcal{W}_u$ on a graph G (weighted or unweighted), suppose T satisfies* (10)*. Let vertex $v \in V$ be such that $T\pi_v = o(1)$, and $\pi_v < 1/2$, then*

$$\mathbf{H}[\pi, v] = (1 + o(1)) \frac{R_v(T)}{\pi_v}. \tag{13}$$

Let $\boldsymbol{A}_t(v)$ denote the event that $\mathcal{W}_u$ does not visit v in steps $0, ..., t$. We next derive a crude upper bound for $\mathbf{Pr}(\boldsymbol{A}_t(v))$ in terms of $\mathbf{H}[\pi, v]$.

Lemma 5. *For a random walk $\mathcal{W}_u$ on a graph G (weighted or unweighted), suppose T satisfies* (10)*, then*

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) \leq \exp\left(\frac{-(1 - o(1)) \lfloor t/\tau_v \rfloor}{2} \right),$$

where $\tau_v = T + 2\mathbf{H}[\pi, v]$.

In order to apply Lemma 5, we shall need to show that the conditions of Lemma 4 are satisfied, and we will need to bound R_v. We start off by bounding the stationary distribution:

Lemma 6. *For a vertex u,*

$$\frac{1}{2n} \leq \pi_u \leq \frac{d(u)}{n}. \tag{14}$$

Corollary 4. $T\pi_u = o(1)$.

We see that for nice degree sequences, the conditions of Lemma 4 are satisfied. It remains to bound R_v, the expected number of returns in the mixing time.

5.1 The number of returns in the mixing time

Let Γ_v denote the subgraph of G induced by all vertices within distance ω of v. From [1], and in conjunction with the restriction $\Delta \leq \omega^{1/4}$, we have the following, which we shall consider to be a typical property:

Proposition 1 ([1]). *With high probability, Γ_v is either a tree or has a unique cycle.*

Let Γ_v° be the set of vertices in Γ_v that are at distance ω from v.

Lemma 7. *Suppose G_w is typical and weighted with the min-degree scheme. Let $\mathcal{W}_v^*$ denote the (weighted) walk on Γ_v starting at v with Γ_v° made into absorbing states. Assume further that there are no cycles in Γ_v°. Let $R_v^* = \sum_{t=0}^{\infty} r_t^*$ where r_t^* is the probability that $\mathcal{W}_v^*$ is at vertex v at time t. Then*

$$R_v = R_v^* + O(\sqrt{\omega}e^{-\Omega(\sqrt{\omega})}).$$

We apply this in the proof of the following lemma:

Lemma 8. *Suppose $\Delta \leq \omega^{1/4}$, G_w is typical and weighted with the min-degree scheme. Let v be a vertex in G_w.*

(a) *If Γ_v is a tree, $R_v \leq \frac{d-1}{d-2} + O(\sqrt{\omega}e^{-\Omega(\sqrt{\omega})})$.*

(b) $R_v \leq \frac{d}{d-2}\frac{d-1}{d-2} + O(\sqrt{\omega}e^{-\Omega(\sqrt{\omega})}) \leq 6 + O(\sqrt{\omega}e^{-\Omega(\sqrt{\omega})})$.

5.2 The number of vertices not locally tree-like

We wish to bound the number of vertices v that are not locally tree-like, i.e., for which Γ_v has a cycle.

Lemma 9. *Suppose $G \in \mathcal{G}(\mathbf{d})$ is drawn* **uar**. *With probability at least $1-n^{-\Omega(1)}$, the number of vertices not locally tree-like is at most $n^{1/10}$.*

6 Upper bound on the cover time

Let the random variable c_u be the time taken by the (weighted) random walk $\mathcal{W}_u$ starting from vertex u to visit every vertex of a connected (weighted) graph G. Let U_t be the number of vertices of G which have not been visited by $\mathcal{W}_u$ by step t. We note the following:

$$\mathbf{COV}_u[G] = \mathbf{E}[c_u] = \sum_{t>0} \mathbf{Pr}(c_u \geq t), \tag{15}$$

$$\mathbf{Pr}(c_u \geq t) = \mathbf{Pr}(c_u > t-1) = \mathbf{Pr}(U_{t-1} > 0) \leq \min\{1, \mathbf{E}[U_{t-1}]\}. \tag{16}$$

Recall that $\boldsymbol{A}_s(v)$ is the event that vertex v has not been visited by time s. It follows from (15), (16) that

$$\mathbf{COV}_u[G] \leq t + 1 + \sum_{s \geq t} \mathbf{E}[U_s] = t + 1 + \sum_{v} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)). \tag{17}$$

We use Lemmas 4 and 5, which hold for weighted random walks (see Chapter 2, General Markov Chains, in [2] for justification of (11) and the inequality $D(s+t) \leq 2D(s)D(t)$. All other expressions in the proofs hold for weighted random walks). Thus,

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) \leq \exp\left(\frac{-(1-o(1))\lfloor t/\tau_v \rfloor}{2} \right),$$

where $\tau_v = T + 2\mathbf{H}[\pi, v]$ and $\mathbf{H}[\pi, v] = (1 + o(1))R_v/\pi_v$.

Hence, for a given v,

$$\begin{aligned}
\sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) &\leq \sum_{s \geq t} \exp\left(\frac{-(1-o(1))\lfloor s/\tau_v \rfloor}{2} \right) \\
&\leq \tau_v \sum_{s \geq \lfloor t/\tau_v \rfloor} \exp\left(\frac{-(1-o(1))s}{2} \right) \\
&\leq 3\tau_v \exp\left(\frac{-(1-o(1))\lfloor t/\tau_v \rfloor}{2} \right) \\
&= 3\tau_v \exp\left(\frac{-(1-o(1))}{2} \left\lfloor \frac{t\pi_v}{T\pi_v + (1+o(1))2R_v} \right\rfloor \right).
\end{aligned}$$

Since $T\pi_v = o(1)$ and $\pi_v \geq 1/2n$ from (14), we get

$$\sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) \leq 3\tau_v \exp\left(\frac{-(1-o(1))}{2} \left\lfloor \frac{t}{(1+o(1))4nR_v} \right\rfloor \right)$$

Let $t = t^* = (1+\epsilon)8\frac{d-1}{d-2} n \log n$ where $\epsilon \to 0$ sufficiently slowly. Then

$$\sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) \leq 3\tau_v \exp\left(-(1 + \Theta(\epsilon)) \frac{d-1}{d-2} \frac{\log n}{R_v} \right) \tag{18}$$

We partition the double sum $\sum_v \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v))$ from (17) into

$$\sum_{v \in V_A} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) + \sum_{v \in V_B} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v))$$

where V_A are locally tree-like and V_B are not.

If v is locally tree-like, then using Theorem 8 **(a)**, the RHS of (18) is bounded by

$$\begin{aligned} 3\tau_v n^{-(1+\Theta(\epsilon))} &= 3(T + 2(1+o(1))R_v/\pi_v) n^{-(1+\Theta(\epsilon))} \\ &\leq (1+o(1)) 12 n R_v n^{-(1+\Theta(\epsilon))} \\ &= O(1) n^{-\Theta(\epsilon)}. \end{aligned}$$

Thus,

$$\sum_{v \in V_A} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) \leq O(1) n^{1-\Theta(\epsilon)} = o(t). \tag{19}$$

For any v (i.e., including those not locally tree-like), (18) is bounded by

$$3\tau_v n^{-(1+\Theta(\epsilon))\frac{d-1}{6(d-2)}} \leq O(1) n^{1-(1+\Theta(\epsilon))\frac{d-1}{\cdot}6(d-2)} \tag{20}$$

Using Lemma 9 to sum the bound (20) over all non locally tree-like vertices, we get

$$\sum_{v \in V_B} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s(v)) \leq O(1) n^{\frac{1}{10}+1-(1+\Theta(\epsilon))\frac{d-1}{6(d-2)}} = O(n^{\frac{1}{2}}) = o(t). \tag{21}$$

Hence, combining (17), (19) and (21) for $t = t^*$, Theorem 3 follows.

Compare this with (3), we see that the speed up,

$$\mathcal{S} = \frac{\mathbf{COV}[G]}{\mathbf{COV}[G_w]} = \Omega(\theta),$$

Therefore $\mathcal{S} \to \infty$ as $n \to \infty$ if $\theta \to \infty$ as $n \to \infty$. That is, we can have an unbounded speed up.

We conjecture that the following tighter bound holds:

Conjecture 2. Equation (4) can be replaced by

$$\mathbf{COV}[G_w] \leq (1+o(1)) \frac{d-1}{d-2}\, n \log n.$$

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7 Appendix

Proof (Proof of Lemma 2). First note that, for $(u, v) \in E$, we have

$$w(u,v) \leq \frac{1}{d(u)} + \frac{1}{d(v)} \leq 2w(u,v)$$

In addition, we have $\sum_{u \in V} \sum_{v \in N(u)} (1/d(u)) = n$. Hence, for undirected graphs,

$$\sum_{u \in V} \sum_{v \in N(u)} \left(\frac{1}{d(u)} + \frac{1}{d(v)} \right) = 2n$$

which concludes the proof.

Proof (Proof of Theorem 1).

A link between electrical network theory and random walks is made via the *commute* time $\mathbf{COM}[u,v] = \mathbf{H}[u,v] + \mathbf{H}[v,u]$ (note that, generally, $\mathbf{H}[u,v] \neq \mathbf{H}[v,u]$). For a graph $G = (V, E, w)$, define $w(G) = \sum_{u \in V} w(u) = 2 \sum_{e \in E} w(e)$. From [4] we have the following lemma:

Lemma 10 ([4]). *Let $G = (V, E, w)$. For any pair of vertices $u, v, \in V$,*

$$\mathbf{COM}[u,v] = w(G) R(u,v),$$

where $R(u,v)$ is the effective resistance between vertices u and v.

We refer the reader to .e.g., [5] for an explanation of effective resistance. For our purposes, it suffices to say that the effective resistance between vertices u and v is at most the resistance of an edge between them. That is, $R(u,v) \leq r(u,v)$, where $r(u,v) = 1/w(u,v)$.

Let $\rho = (x_0, x_1, \ldots, x_\ell)$ where $x_0 = u$ and $x_\ell = v$ be a shortest path between u and v.

$$\begin{aligned} \mathbf{H}[u,v] &\leq \sum_{i=0}^{\ell-1} \mathbf{H}[x_i, x_{i+1}] \\ &\leq \sum_{i=0}^{\ell-1} \mathbf{COM}[x_i, x_{i+1}] \qquad (22) \\ &= w(G) \sum_{i=0}^{\ell-1} R(x_i, x_{i+1}). \qquad (23) \end{aligned}$$

Since $R(x,y) \leq r(x,y) = \min\{d(x), d(y)\}$, we have

$$\sum_{i=0}^{\ell-1} R(x_i, x_{i+1}) \leq \sum_{i=0}^{\ell-1} \min\{d(x_i), d(x_{i+1})\} \leq \sum_{i=0}^{\ell-1} d(x_i) \leq 3n,$$

where the last inequality follows by Lemma 1. By (2) we have $w(G) \leq 2n$, and the theorem follows.

Proof (Proof of Lemma 3). Since $w(u) = \sum_{v \in V} w(u,v)$ (where $w(u,v) = 0$ if $(u,v) \notin E$) and $\pi(u) = \frac{w(u)}{w(G)}$ and $P_{u,v} = \frac{w(u,v)}{w(u)}$ we have $Q(u,v) = \pi(u) P_{u,v} = \frac{w(u,v)}{w(G_w)}$ and

$$Q(S, S') = \frac{1}{w(G_w)} \sum_{u \in S, v \in S'} w(u,v).$$

Since

$$\pi(S) = \sum_{u \in S} \pi(u) = \frac{1}{w(G_w)} \sum_{u \in S} w(u),$$

we have

$$\Phi(G_w) = \min_{\pi(S) \le 1/2} \frac{\sum_{u \in S, v \in S'} w(u,v)}{\sum_{u \in S} w(u)}. \tag{24}$$

In [1, Lemma 7], it is shown that the conductance of the simple random walk is bounded below by $\varepsilon = 1/100$, i.e., for a graph G picked **uar** from $\mathcal{G}(\mathbf{d}_n)$, subject to $(\mathbf{d}_n)_n$ being nice, **whp**

$$\mathcal{E}(S) = \frac{|E(S : \overline{S})|}{d(S)} \ge \varepsilon \tag{25}$$

for any set S such that $\pi(S) \le 1/2$, where $E(S : \overline{S})$ is the set of edges with one end in S and the other in $\overline{S}$, and $d(S) = \sum_{v \in S} d(v)$. This implied that for an unweighted (or uniformly weighted) graph, $\Phi(G) \ge \varepsilon$, equation (24) becomes

$$\Phi(G) = \min_{\pi(S) \le 1/2} \frac{|E(S : \overline{S})|}{d(S)}.$$

If Δ is the maximum degree in $\mathbf{d}$, then $w(e) \ge 1/\Delta$ for any edge e. Therefore, $\Phi(G_w) \ge \Phi(G)/\Delta \ge 1/(100\Delta)$.

Proof (Proof of Corollary 3). Combining Lemma 3 and (5) for a lazy walk, we have

$$|P_u^{(t)}(x) - \pi_x| \le \left(\frac{\pi_x}{\pi_u}\right)^{1/2} \left(1 - \frac{\Phi^2}{2}\right)^t \le \Delta^{1/2} \left(1 - \frac{1}{K\Delta^2}\right)^T \le \omega^{1/8} \exp\left(-\frac{\omega^2 \log n}{K\omega^{1/2}}\right) \le n^{-3}$$

where K is a constant.

Proof (Proof of Lemma 4). Let $D(t) = \max_{u,x} |P_u^{(t)}(x) - \pi_x|$. It follows from e.g. [2] that $D(s+t) \le 2D(s)D(t)$. Hence, since $\max_{u,x} |P_u^{(T)}(x) - \pi_x| \le \pi_v$, then for each $k \ge 1$, $\max_{u,x} |P_u^{(kT)}(x) - \pi_x| \le (2\pi_v)^k$. Thus

$$\begin{aligned} Z_{v,v} &= \sum_{t=0}^{\infty} (P_v^{(t)}(v) - \pi_v) \\ &\le \sum_{t<T} (P_v^{(t)}(v) - \pi_v) + T \sum_{k \ge 1} (2\pi_v)^k \\ &= R_v(T) - T\pi_v + O(T\pi_v) \\ &= R_v(T)(1 + o(1)). \end{aligned}$$

The last inequality follows because $R_v(T) \ge 1$.

Proof (Proof of Lemma 5). Let $\rho = \rho(G, T, u)$ be the distribution of $\mathcal{W}_u$ on G after T, then

$$\mathbf{H}[\rho, v] = (1 + o(1))\mathbf{H}[\pi, v].$$

Indeed,

$$\begin{aligned}\mathbf{H}[\rho, v] &= \sum_{w \in V} \rho_w \mathbf{H}[w, v] \\ &= (1 + o(1)) \sum_{w \in V} \pi_w \mathbf{H}[w, v] \\ &= (1 + o(1))\mathbf{H}[\pi, v]. \end{aligned} \tag{26}$$

Let $h_\rho(v)$ be the time to hit v starting from ρ. Then $\mathbf{E}[h_\rho(v)] = \mathbf{H}[\rho, v]$ so by Markov's inequality and using (26),

$$\mathbf{Pr}(h_\rho(v) \geq 2\mathbf{H}[\pi, v]) \leq \frac{(1 + o(1))}{2}.$$

By restarting the process at $\mathcal{W}_u(0) = u, \mathcal{W}_u(\tau_v), \mathcal{W}_u(2\tau_v), ..., \mathcal{W}_u(\lfloor t/\tau_v \rfloor \tau_v)$ we obtain

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) \leq \left(\frac{(1 + o(1))}{2}\right)^{\lfloor t/\tau_v \rfloor}.$$

Proof (Proof of Lemma 6). $\pi_u = \frac{w(u)}{w(G)}$. Recalling $N(u)$ is the neighbour set of a vertex u, use

$$\sum_{v \in N(u)} w(u, v) \geq \sum_{v \in N(u)} \frac{1}{d(u)} = 1$$

and

$$\sum_{v \in N(u)} w(u, v) \leq \sum_{v \in N(u)} 1 = d(u)$$

with (2).

Proof (Proof of Lemma 7). Observe,

$$R_v - R_v^* = \left(\sum_{t=0}^{\omega} r_t - r_t^*\right) + \left(\sum_{t=\omega+1}^{T} r_t - r_t^*\right) - \sum_{t=T+1}^{\infty} r_t^*.$$

Case $t \leq \omega$. For $t \leq \omega$, $r_t^* = r_t$. Thus we can write

$$\left(\sum_{t=0}^{\omega} r_t - r_t^*\right) = 0. \tag{27}$$

Case $\omega + 1 \le t \le T$. We use (7) with $u = v$. We have $z = (1 - \Phi(G_w)^2/2) < (1 - 1/K\sqrt{\omega})$ for some constant K. Observe $T\pi_v = O(1/n^c)$ for some constant $c > 0$, so $T\pi_v = o(\sqrt{\omega}e^{-\sqrt{\omega}})$. Hence,

$$\sum_{t=\omega+1}^{T} |r_t - r_t^*| = \sum_{t=\omega+1}^{T} r_t \le \sum_{t=\omega+1}^{T} (\pi_v + z^t) \le T\pi_v + \frac{z^\omega}{1-z} = O(\sqrt{\omega}e^{-\Omega(\sqrt{\omega})}). \tag{28}$$

Case $t \ge T + 1$. It remains to estimate $\sum_{t=T+1}^{\infty} r_t^*$.

Let $x \in \Gamma_v$. Let p be the probability of movement away from v and q the probability toward it. Observe the min-degree weighting scheme gives

$$\frac{p}{q} \ge \frac{(d_x - 1)1/d_x}{1/d} \ge d\frac{d-1}{d} = d - 1.$$

Hence, the probability of moving away from v for the weighted walk is at least the probability of the same for an unweighted walk on a d-regular tree. Let σ_t be the probability that such a walk has not been absorbed at time t, where the abosrbing vertices are those at distance ω from the root. Then $r_t^* \le \sigma_t$, and so

$$\sum_{t=T+1}^{\infty} r_t^* \le \sum_{t=T+1}^{\infty} \sigma_t.$$

We estimate an upper bound for σ_t as follows: Consider an unbiased random walk $X_0^{(b)}, X_1^{(b)}, \ldots$ starting at $|b| < a \le \omega$ on the finite line $(-a, -a+1, ..., 0, 1, ..., a)$, with absorbing states $-a, a$. $X_m^{(0)}$ is the sum of m independent ± 1 random variables. The central limit theorem implies that there exists a constant $c > 0$ such that

$$\mathbf{Pr}(X_{ca^2}^{(0)} \ge a \text{ or } X_{ca^2}^{(0)} \le -a) \ge 1 - e^{-1/2}.$$

Now for any t and b with $|b| < a$, we have

$$\mathbf{Pr}(|X_t^{(b)}| < a) \le \mathbf{Pr}(|X_t^{(0)}| < a)$$

which is justified with the following game: We have two walks, A and B coupled to each other, with A starting at position 0 and B at position b, which, w.l.o.g, we shall assume is positive. The walk is a simple random walk which comes to a halt when either of the walks hits an absorbing state (that being, $-a$ or a). Since they are coupled, B will win iff they drift $(a - b)$ to the right from 0 and A will win iff they drift $-a$ to the left from 0. Therefore, given the symmetry of the walk, B has a higher chance of winning.
Thus

$$\mathbf{Pr}(|X_{2t}^{(0)}| < a) \le \mathbf{Pr}(|X_t^{(0)}| < a)^2,$$

since after t steps, the worst case position for the walk to be at is the origin, 0. Consequently, for any b with $|b| < a$,

$$\mathbf{Pr}(|X^{(b)}_{2ca^2}| \geq a) \geq 1 - e^{-1}. \tag{29}$$

Hence, for $t > 0$,

$$\sigma_t = \mathbf{Pr}(|X^{(0)}_\tau| < a,\, \tau = 0, 1, \ldots, t) \leq e^{-\lfloor t/(2ca^2) \rfloor}. \tag{30}$$

Thus

$$\sum_{t=T+1}^{\infty} \sigma_t \leq \sum_{t=T+1}^{\infty} e^{-t/(3c\omega^2)} \leq \frac{e^{-T/(3c\omega^2)}}{1 - e^{-1/(3c\omega^2)}} = O(\omega^2/n^{\Omega(1)}) = O(\sqrt{\omega} e^{-\Omega(\sqrt{\omega})}). \tag{31}$$

Proof (Proof of Lemma 8). **(a)** As per proof of Case $t \geq T + 1$ in Lemma 7, we can bound the number of returns by considering the unweighted walk on a tree in which all internal vertices, except possibly the root, have degree d. The leaves of the tree are absorbing states, and we apply Lemma 7.

For a biased random walk on $(0, 1, ..., k)$, starting at vertex 1, with absorbing states $0, k$, and with transition probabilities at vertices $(1, \ldots, k-1)$ of $q = \mathbf{Pr}(\text{move left})$, $p = \mathbf{Pr}(\text{move right})$; then

$$\mathbf{Pr}(\text{absorption at } k) = \frac{(q/p) - 1}{(q/p)^k - 1}. \tag{32}$$

This is the escape probability ρ - the probability that after the particle moves from v to an adjacent vertex, it reaches an absorbing state without having visited v again. $R^*_v = 1 + 1/\rho - 1 = 1/\rho$.
We project $\mathcal{W}^*_v$ onto $(0, 1, \ldots, \omega)$ with $p = \frac{d-1}{d}$ and $q = \frac{1}{d}$ giving

$$R^*_v \leq \left(1 - \frac{1}{(d-1)^\omega}\right) \frac{d-1}{d-2} = \frac{d-1}{d-2} - O\left((d-1)^{-\omega}\right) \tag{33}$$

and part **(a)** of the lemma follows.

(b) Since any cycle in Γ_v is unique, at most two of the edges, $(v, u_1), (v, u_2)$ out of v lead to vertices on a cycle. Let *bad* be the event event of moving from v to some $u \in \{u_1, u_2\}$ Then

$$\mathbf{Pr}(bad) \leq \frac{2/d}{2/d + (d_v - 2)(1/d_v)} = \frac{2/d}{2/d + 1 - 2/d_v} \leq 2/d.$$

Denoting the neighbour set of v by $N(v)$, the probability of moving from v to some $u \in N(v) \setminus \{u_1, u_2\}$ is then at least $\frac{d-2}{d}$, implying $\frac{d}{d-2}$ returns to v in expectation for every transition from v to $N(v) \setminus \{u_1, u_2\}$. Assuming that a move from v to u_1 or u_2 always results in an immediate return, we can bound $R_v \leq \frac{d}{d-2}(\frac{d-1}{d-2} + O(\sqrt{\omega} e^{-\Omega(\sqrt{\omega})})) \leq 6 + O(\sqrt{\omega} e^{-\Omega(\sqrt{\omega})})$.

Proof (Proof of Lemma 9). Recall θ is the average degree. Letting $\mathcal{F}(2x) = \frac{(2x)!}{2^x x!}$, it can be shown that for integer $x > 0$, $\frac{\mathcal{F}(\theta n - 2x)}{\mathcal{F}(\theta n)} \leq \left(\frac{1}{\theta n - 2x}\right)^x$.

The expected number of small cycles has upper bound

$$\begin{aligned}
&\sum_{k=3}^{2\omega+1} \binom{n}{k} \frac{(k-1)!}{2} (\Delta(\Delta-1))^k \frac{\mathcal{F}(\theta n - 2k)}{\mathcal{F}(\theta n)} \\
&\leq \sum_{k=3}^{2\omega+1} n^k \Delta^{2k} \left(\frac{1}{\theta n - 2k}\right)^k \\
&\leq \sum_{k=3}^{2\omega+1} \Delta^{2k} \left(\frac{n}{\theta n - 4\omega - 2}\right)^k \\
&\leq \Delta^{4(\omega+1)}.
\end{aligned}$$

Therefore, the expected number of vertices within distance ω of a cycle is at most $\Delta^{4(\omega+1)}\Delta^{\omega} = \Delta^{5\omega+4}$. Since $\Delta \leq \omega^{1/4}$, the lemma follows by Markov's inequality.